ECON 301 - PS #1

(REC #1)

Problem 8:

a) $Q_{dt} = \beta_0 + \beta_1 P_{dt} + U_t$

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{T} p_t q_t}{\sum_{t=1}^{T} p_t^2} \qquad \left(\sum Q_d = 1004.9 \text{ and } \sum P_d = 974\right)$$

With $T = 10$, $\bar{Q}_d = \frac{\sum Q_d}{T} = 100.49$, $\bar{P}_d = \frac{\sum P_d}{T} = 97.4$

$$(Q_t - \bar{Q}) = q_t = \begin{bmatrix} -2.89 \\ -3.29 \\ -3.19 \\ -4.49 \\ -1.29 \\ -0.19 \\ -0.19 \\ 3.61 \\ 4.81 \\ 7.11 \end{bmatrix} \qquad (P_t - \bar{P}) = p_t = \begin{bmatrix} -8.8 \\ -6.4 \\ 0.5 \\ 4.9 \\ 4.8 \\ 5.1 \\ -0.4 \\ -1.6 \\ -1 \\ 2.9 \end{bmatrix} \qquad p_t q_t = \begin{bmatrix} 25.432 \\ 21.056 \\ -1.595 \\ -22.001 \\ -6.192 \\ -0.969 \\ 0.076 \\ -5.776 \\ -4.81 \\ 20.619 \end{bmatrix}$$

$\sum p_t^2 = 203.84 \qquad \sum p_t q_t = 25.84$

(Careful: $\sum p_t^2 \neq \left(\sum p_t\right)^2$)

$$\hat{\beta}_1 = \frac{\sum p_t q_t}{\sum p_t^2} = \frac{25.84}{203.84} \cong 0.12$$

$\hat{\beta}_0 = \bar{Q} - \bar{P}\hat{\beta}_1 \cong 88.14$ ($\bar{Q}$ and $\bar{P}$ was found above)

$\underline{\hat{Q}_{dt} = 88.14 + 0.12 P_{dt}}$

b) s.e. of $\hat{\beta} = \sqrt{Var(\hat{\beta})}$, $\left(\hat{Var}(\hat{\beta}_0) = \hat{\sigma}^2 \dfrac{\sum P_t^2}{T \sum p_t^2} \begin{matrix} \to \text{capital } P \\ \to \text{small } p \end{matrix} \right.$

$\left. \hat{Var}(\hat{\beta}_1) = \dfrac{\hat{\sigma}^2}{\sum p_t^2} \to \text{small } p \right)$

We first need to find $\hat{\sigma}^2$, which is $\dfrac{\sum \hat{u}_t^2}{T-k-1}$

$\hat{u}_t$ is obtained from $Q_t - \hat{Q}_t$, where $\hat{Q}_t = \hat{\beta}_0 + \hat{\beta}_1 P_t = 88.14 + 0.12 P_t$

From our data we find $\sum \hat{u}_t^2 = 134.69$, $T = 10$ and $k = 1$

So $\hat{\sigma}^2 = \dfrac{134.69}{10-1-1} \cong 16.83$

Therefore, $\hat{Var}(\hat{\beta}_1) = \dfrac{\hat{\sigma}^2}{\sum p_t^2} = \dfrac{16.83}{203.84} \cong 0.08$

$se(\hat{\beta}_1) = \sqrt{\hat{Var}(\hat{\beta}_1)} = \sqrt{0.08} \cong 0.28$

$\hat{Var}(\hat{\beta}_0) = \hat{\sigma}^2 \dfrac{\sum P_t^2}{T \sum p_t^2} = \underset{16.83}{16.83} \times \dfrac{95071.4}{10 \times 203.8} \cong 785$

$se(\hat{\beta}_0) = \sqrt{Var(\hat{\beta}_0)} = \sqrt{785} \cong 28$

---

c) $r_{QP} = \dfrac{\sum qp}{\sqrt{\sum q^2 \sum p^2}} = \dfrac{25.84}{\sqrt{138 \times 203.84}} \cong \dfrac{25.84}{167.71} \cong 0.15$

---

d) $R^2 = 1 - \dfrac{\sum \hat{u}_t^2}{\sum q_t^2}$ or $R^2 = r_{QP}^2 \cong 0.02$

---

e) Price elasticity of demand $= \dfrac{dQ}{dP} \cdot \dfrac{P}{Q}$

Since $\hat{Q}_t = 88.14 + 0.12 P_t$ $\dfrac{dQ}{dP} = 0.12$

For 1989 $\Rightarrow$ $0.12 \times \dfrac{95.8}{104.1} \cong 0.11$ where $95.8 = P_{1989}$, $104.1 = Q_{1989}$

For 1991 $\Rightarrow$ $0.12 \times \dfrac{100.3}{107.6} \cong 0.11$ $(100.3 = P_{91}$, $107.6 = Q_{91})$

We could also use $\hat{Q}$ instead of Q as respective Q values.

Problem 11 ;

(I) $Y_t = \beta_0 + \beta_1 X_t + u_t$

(II) $Y_t = \alpha_0 + \alpha_1 (X_t - \bar{X}) + v_t$

a) Define $x_t = (X_t - \bar{X})$

$$\hat{\beta}_1 = \frac{\sum x_t y_t}{\sum x_t^2} \quad , \quad \hat{\beta}_0 = \bar{Y} - \bar{X}\hat{\beta}_1$$

$$\hat{\alpha}_1 = \frac{\sum (x_t - \bar{x}_t) y_t}{\sum (x_t - \bar{x}_t)^2} \Rightarrow \bar{x}_t = \frac{\sum x_t}{N} = 0 \quad (\sum x_t = 0)$$

Therefore $\hat{\alpha}_1 = \dfrac{\sum x_t y_t}{\sum x_t^2} = \hat{\beta}_1$

b) $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

$$\hat{\alpha}_0 = \bar{Y} - \hat{\alpha}_1 \underbrace{\bar{x}_t}_{\bar{x}_t = 0 \Rightarrow 0} = \bar{Y}$$

Problem 13

Since OLS estimators are obtained by minimizing SSE, we have:

$$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}} = 0 \quad , \quad \hat{u}_t = Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_t$$

FOC's:

1) $$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}_0} = -2 \sum (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_t) = -2 \sum \hat{u}_t = 0$$

2) $$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}_1} = -2 \sum \hat{u}_t X_t = 0$$

Sample covariance:

$$Cov(X_t, \hat{u}_t) = \frac{1}{N-1} \sum (X_t - \bar{X})(\hat{u}_t - \underbrace{\bar{\hat{u}}_t}_{=0 \text{ since } \sum \hat{u}_t = 0 \text{ from FOC (1)}})$$

$$= \frac{1}{N-1} \left[\underbrace{\sum X_t \hat{u}_t}_{=0 \text{ from (2)}} - \bar{X} \underbrace{\sum \hat{u}_t}_{=0 \text{ from (1)}} \right] = 0$$

Page 4

Problem 18

a) Minimize $\sum \hat{u}_t^2$ w.r.t $\hat{\beta}$, with $\hat{u}_t^2 = (Y_t - \hat{\beta} X_t)^2$

$$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}} = -\cancel{2} \sum (Y_t - \hat{\beta} X_t) X_t = 0$$

$$\Rightarrow \sum X_t Y_t = \hat{\beta} \sum X_t^2 \Rightarrow \hat{\beta} = \frac{\sum X_t Y_t}{\sum X_t^2}$$

b) $Var(\hat{\beta}) = E[\hat{\beta} - E(\hat{\beta})]^2$

- what is $E(\hat{\beta})$?

$$\hat{\beta} = \frac{\sum X_t \underbrace{(\beta X_t + u_t)}_{Y_t}}{\sum X_t^2} = \beta \frac{\cancel{\sum X_t^2}}{\cancel{\sum X_t^2}} + \frac{\sum X_t u_t}{\sum X_t^2} = \beta + \frac{\sum X_t u_t}{\sum X_t^2}$$

$$E(\hat{\beta}) = \beta + \underbrace{\frac{\sum X_t E(u_t)}{\sum X_t^2}}_{0} = \beta$$

- $Var(\hat{\beta}) = E[\hat{\beta} - \beta]^2 = E\left[\frac{\sum X_t u_t}{\sum X_t^2}\right]^2$

Note that $(\sum X_t u_t)^2 = X_1^2 u_1^2 + X_2^2 u_2^2 + \dots + X_T^2 u_T^2 + \underbrace{X_1 u_1 u_2 X_2 + \dots}_{\text{cross products}}$

$$E\left[\sum X_t u_t\right]^2 = E\left[X_1^2 u_1^2 + X_2^2 u_2^2 + \dots + X_T^2 u_T^2\right] + E\left[\underbrace{X_1 u_1 u_2 X_2}_{E(u_1 u_2) = 0} + \dots\right]$$

Since $E(u_t, u_{t-j}) = 0$, $E(\text{cross products}) = 0$

So $E\left[\sum X_t u_t\right]^2 = E\left[\sum X_t^2 u_t^2\right]$

$$E\left[\frac{\sum X_t u_t}{\sum X_t^2}\right]^2 = E\left[\frac{\sum X_t^2 u_t^2}{(\sum X_t^2)^2}\right] = \frac{\cancel{\sum X_t^2} E(u_t^2)}{(\sum X_t^2)^{\cancel{2}}} \rightarrow E(u_t^2) = \sigma^2 \text{ is a constant}$$

$$Var(\hat{\beta}) = \sigma^2 \frac{1}{\sum X_t^2} = \frac{\sum u_t^2 / N-1}{\sum X_t^2}$$

In $\hat{Var}$ we use $\hat{\sigma}^2$, which is $\frac{\sum \hat{u}_t^2}{N-k-1}$ → with no intercept this is $N-k$ $(k=1)$

Therefore, $\hat{Var}(\hat{\beta}) = \frac{\sum \hat{u}_t^2}{N-1} \cdot \frac{1}{\sum X_t^2}$

c) When there is an intercept term in a simple regression, we had the following conditions for min SSR:

1) $$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}_0} = -2 \sum \hat{u}_t = 0$$

2) $$\frac{\partial \sum \hat{u}_t^2}{\partial \hat{\beta}_1} = -2 \sum \hat{u}_t X_t = 0$$

When there is no intercept term, we only have the second condition. Therefore we do not <u>necessarily</u> have $\sum \hat{u}_t = 0$ in OLS.

Problem 23

(a),(b),(c) together:

$$Y_t = \beta_0 + \beta_1 X_t + u_t$$

$X_t^* = cX_t$ ($c=5$ in this question)

Also $\hat{\alpha}_1 = \frac{\sum x_t y_t}{\sum x_t^2}$, $\hat{\alpha}_0 = \overline{Y} - \hat{\alpha}_1 \overline{X}$

$\overline{X^*}$ then becomes $\frac{\sum cX_t}{N} = c\frac{\sum X_t}{N} = c\overline{X}$

$$x_t^* = X_t^* - \overline{X^*} = cX_t - c\overline{X} = cx_t$$

$$\hat{\beta}_1 = \frac{\sum y_t x_t^*}{\sum x_t^{*2}} = \frac{c\sum y_t x_t}{c^2 \sum x_t^2} = \frac{1}{c}\hat{\alpha}_1$$

$\hat{\beta}_0$ then becomes $\Rightarrow \overline{Y} - \hat{\beta}_1 \overline{X^*} = \overline{Y} - \frac{1}{\cancel{c}}\hat{\alpha}_1 \cancel{c}\overline{X} = \overline{Y} - \hat{\alpha}_1 \overline{X} = \hat{\alpha}_0$

So $\hat{\beta}_1 = \frac{1}{c}\hat{\alpha}_1$ and $\hat{\beta}_0 = \hat{\alpha}_0$

Fitted values:

$\hat{Y}_t^{II} = \hat{\alpha}_0 + \hat{\alpha}_1 X_t$ and $\hat{Y}_t^{I} = \hat{\beta}_0 + \hat{\beta}_1 X_t^*$

$$= \hat{\alpha}_0 + \frac{1}{\cancel{c}}\hat{\alpha}_1 \cancel{c} X_t$$

$$= \hat{\alpha}_0 + \hat{\alpha}_1 X_t$$

$$\hat{u}_t^{II} = Y_t - \hat{Y}_t^{II} = Y_t - \hat{Y}_t^{I} = \hat{u}_t^{I}$$

With same fitted values, residuals are also the same.

Problem 25

a) $Y_t^* = b_1^* + b_2^* X_t + u_t^*$, where $Y_t^* = Y_t - \bar{Y}$ and $X_t^* = X_t - \bar{X}$

(so $Y_t^* = y_t$ and $X_t^* = x_t$ in previous examples' terms)

$$\hat{b}_1^* = \overline{Y_t^*} - \hat{b}_2^* \overline{X_t^*}$$

$$\overline{Y_t^*} = \frac{\sum (Y_t - \bar{Y})}{N} = 0 \quad \left(\frac{\sum Y_t}{N} - \frac{\cancel{N}\bar{Y}}{\cancel{N}} = \bar{Y} - \bar{Y} = 0 \right)$$

$\overline{X_t^*}$ is also 0, (you can also think of it by $\sum x_t = \sum y_t = 0$)

Therefore $\hat{b}_1^* = \overline{Y_t^*} - \hat{b}_2^* \overline{X_t^*} = 0$.

b) $\hat{b}_2^* = \dfrac{\sum y_t^* x_t^*}{\sum x_t^{*2}}$, $\quad y_t^* = Y_t^* - \underbrace{\overline{Y_t^*}}_{0} = Y_t^*$

$$x_t^* = X_t^* - \underbrace{\overline{X_t^*}}_{0} = X_t^*$$

Therefore $\dfrac{\sum y_t^* x_t^*}{\sum x_t^{*2}} = \dfrac{\sum Y_t^* X_t^*}{\sum X_t^{*2}} = \dfrac{\sum (Y_t - \bar{Y})(X_t - \bar{X})}{\sum (X_t - \bar{X})^2} = \hat{\beta}_1$ from $Y_t = \beta_0 + \beta_1 X_t + u_t$ (which is $\dfrac{\sum x_t y_t}{\sum x_t^2}$)

Problem 30:

a) One of the researchers has:

$$1000 Y_t = \alpha + \beta 1000 X_t + \varepsilon_t$$

$\bar{Y}$ and $\bar{X}$ will also be multiplied by 1000, and so will x and y.

i) $\tilde{\beta}$ obtained from this regression will be:

$$\frac{\sum 1000y\, 1000x}{\sum (1000x)^2} = \frac{1000^2 \sum yx}{1000^2 \sum x^2} = \frac{\sum xy}{\sum x^2} = \hat{\beta}$$ that the other researcher gets (from $Y_t = \alpha + \beta X_t + e_t$)

$\tilde{\alpha}$ will be $1000\bar{Y} - \tilde{\beta} 1000 \bar{X} = 1000(\bar{Y} - \hat{\beta}\bar{X}) = 1000\hat{\alpha}$

So $\tilde{\alpha}$ and $\hat{\alpha}$ will be different from each other.

ii) The new residuals will also be different:

$$\tilde{u}_t = 1000 Y_t - 1000\hat{\alpha} - \hat{\beta} 1000 X_t = 1000(Y_t - \hat{Y}_t) = 1000\hat{u}_t$$

- $se(\hat{\beta}) = \sqrt{\frac{\hat{\sigma}^2}{\sum x_t^2}}$ ($\nearrow \sqrt{Var(\hat{\beta})}$), $\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{N-1}$

- t statistic of $\tilde{\beta}$ depends negatively on its standard error, given as:

$$se(\tilde{\beta}) = \sqrt{Var(\tilde{\beta})} = \sqrt{\frac{\tilde{\sigma}^2}{\sum (1000x)^2}}, \quad \tilde{\sigma}^2 = \frac{\sum \tilde{u}_t^2}{T-k-1} = \frac{1000^2 \sum \hat{u}_t^2}{T-k-1} = 1000^2 \hat{\sigma}^2$$

$$se(\tilde{\beta}) = \sqrt{\frac{\cancel{1000^2}\hat{\sigma}^2}{\cancel{1000^2}\sum x_t^2}} = se(\hat{\beta})$$

$\tilde{\beta} = \hat{\beta}$ and $se(\hat{\beta}) = se(\tilde{\beta})$, therefore $\underbrace{\frac{\tilde{\beta}}{se(\tilde{\beta})}}_{t_{\tilde{\beta}}} = \underbrace{\frac{\hat{\beta}}{se(\hat{\beta})}}_{t_{\hat{\beta}}}$

iii) Remember that both y_t and u_t are multiplied by 1000.

$$R^2 = 1 - \frac{\sum \tilde{u}_t^2}{\sum 1000^2 y_t^2} = 1 - \frac{\cancel{1000^2} \sum \hat{u}_t^2}{\cancel{1000^2} \sum y_t^2} = 1 - \frac{\sum \hat{u}_t^2}{\sum y_t^2}$$

Therefore R^2 remains the same.

b) One of the researchers has (Researcher B):

$$\frac{1}{1000} Y_t = \alpha + \beta X_t + \varepsilon_t$$

x_t is same with y_t multiplied by $\frac{1}{1000}$.

i) $\tilde{\beta} = \frac{\sum \frac{1}{1000} y x}{\sum x^2} = \frac{1}{1000}\hat{\beta}$

$$\tilde{\alpha} = \frac{1}{1000}\bar{Y} - \frac{1}{1000}\hat{\beta}\bar{X} = \frac{1}{1000}(\bar{Y} - \hat{\beta}\bar{X}) = \frac{1}{1000}\hat{\alpha}$$

ii) New residuals, $\tilde{u}_t$ will be:

$$\tilde{u}_t = 1000 Y_t - \underbrace{\frac{1}{1000}\hat{\alpha}}_{\tilde{\alpha}} - \underbrace{\frac{1}{1000}\hat{\beta}}_{\tilde{\beta}} X_t = \frac{1}{1000}\hat{u}_t \Rightarrow \tilde{\sigma}^2 = \frac{1}{1000^2}\hat{\sigma}^2$$

(see part (a))

$$se(\tilde{\beta}) = \sqrt{Var(\tilde{\beta})} = \sqrt{\frac{\tilde{\sigma}^2}{\sum x_t^2}} = \sqrt{\frac{\frac{1}{1000^2}\hat{\sigma}^2}{\sum x_t^2}} = \frac{1}{1000} se(\hat{\beta})$$

$$t_{\tilde{\beta}} = \frac{\tilde{\beta}}{se(\tilde{\beta})} = \frac{\cancel{\frac{1}{1000}}\hat{\beta}}{\cancel{\frac{1}{1000}} se(\hat{\beta})} = t_{\hat{\beta}}$$

iii) New R^2, $1 - \dfrac{\frac{1}{\cancel{1000^2}}\sum \hat{u}_t^2}{\frac{1}{\cancel{1000^2}}\sum y_t^2}$ will also be the same.

Problem 33

See problem 18.