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ECON 301 - REC #2
(PS #1)

Problem 9

Y	X
-2	-2
-1	0
0	1
1	0
2	1

$$\Rightarrow \left.\begin{array}{l}\sum X_t = 0 \\ \sum Y_t = 0\end{array}\right\} \quad \begin{array}{l}\bar{X} = 0 \\ \bar{Y} = 0\end{array}$$

a) $y_t = Y_t - \bar{Y} = Y_t, \quad x_t = X_t - \bar{X} = X_t$

$$\hat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2} = \frac{\sum X_t Y_t}{\sum X_t^2} = \frac{6}{6} = 1, \quad \hat{\alpha} = \underbrace{\bar{Y}}_{0} - \underbrace{\hat{\beta}}_{1}\underbrace{\bar{X}}_{0} = 0$$

b) $\hat{Y}_t = \underbrace{\hat{\alpha}}_{0} + \underbrace{\hat{\beta}}_{1} X_t = X_t$

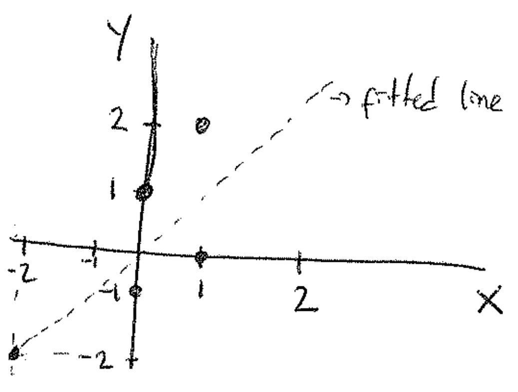
$$\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{\underbrace{N-k-1}_{N-1 \text{ without intercept term}}} = \frac{\sum \hat{u}_t^2}{4}, \quad \hat{u}_t = Y_t - \hat{Y}_t = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \sum \hat{u}_t^2 = 4$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{N-k-1} = \frac{4}{4} = 1$$

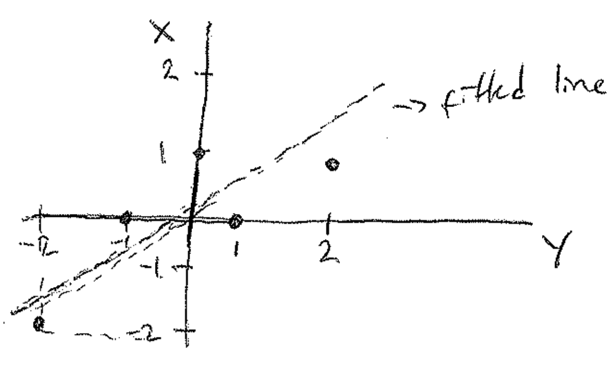
c) $X_t = \gamma + \delta Y_t + v_t$

$$\hat{\delta} = \frac{\sum X_t Y_t}{\sum Y_t^2} = \frac{6}{10} = 0.6, \quad \hat{\gamma} = \underbrace{\bar{X}}_{0} - 0.6\underbrace{\bar{Y}}_{0} = 0$$

d) In the first regression we have $\hat{Y}_t = X_t$
" second regression " $\hat{X}_t = 0.6 Y_t$



Regression I



Regression II

②

We have different fitted regression lines for the two regressions. Since the data points are different for each dependent variable and we try to fit a line so that the total (squared) distance of these points to the fitted line is minimized (sum of the squared residuals is minimized), the resulting characteristics of each fitted line, indicated by the OLS estimators, are different.

27) $R^2 = r_{xy}^2 = r_{yx}^2$

Since the correlation coefficient is symmetric, the resulting R^2 will also be the same and equal to:

$$R^2 = \frac{\left[\sum x_t y_t\right]^2}{\sum x_t^2 \sum y_t^2} \quad \text{where } x_t = X_t - \bar{X} \text{ and } y_t = Y_t - \bar{Y}.$$

28)

a) We have two regressions:

(I) $Y_t = \beta_0 + \beta_1 X_t + u_t$

(II) $X_t = \alpha_0 + \alpha_1 Y_t + u_t$

where $\hat{\beta}_1 = \frac{\sum x_t y_t}{\sum x_t^2}$ and $\hat{\alpha}_1 = \frac{\sum x_t y_t}{\sum y_t^2}$

$$R^2 = r_{xy}^2 = r_{yx}^2 = \frac{\left[\sum x_t y_t\right]^2}{\sum x_t^2 \sum y_t^2}$$

(As shown in problem 27, this is the same for both regressions.)

$$= \underbrace{\frac{\sum x_t y_t}{\sum x_t^2}}_{\hat{\beta}_1} \cdot \underbrace{\frac{\sum x_t y_t}{\sum y_t^2}}_{\hat{\alpha}_1}$$

For $\hat{\beta}_1$ to be equal to $1/\hat{\alpha}_1$, we would therefore need:

$$R^2 = \hat{\beta}_1 \cdot \hat{\alpha}_1 = \frac{1}{\hat{\alpha}_1} \cdot \hat{\alpha}_1 = 1$$

Therefore if $\hat{\beta}_1 = \frac{1}{\hat{\alpha}_1}$, then $R^2 = 1$ must hold. So the two coefficients can be the reciprocals of each other only if $R^2 = 1$.

③

b) We know that $r_{xy} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

Now for Y^* and X^* we will have the following factors:

$$Cov(X^*,Y^*) = E\left[(X^*-E(X^*))(Y^*-E(Y^*))\right], \quad X^* = b_1+b_2X, \quad Y^* = a_1+a_2Y$$

(a and b's are constants)

$$= E\left[(\cancel{b_1}+b_2X-\cancel{b_1}-b_2E(X))(\cancel{a_1}+a_2Y-\cancel{a_1}-a_2E(Y))\right]$$

$$= E\left[b_2(X-E(X))a_2(Y-E(Y))\right] = a_2b_2E\left[(X-E(X))(Y-E(Y))\right]$$

$$= a_2b_2Cov(X,Y)$$

$$Var(Y^*) = Var(a_1+a_2Y) = E\left[\cancel{a_1}+a_2Y-\cancel{a_1}-a_2E(Y)\right]^2$$

$$= E\left[a_2(Y-E(Y))\right]^2$$

$$= a_2^2 E(Y-E(Y))^2$$

$$= a_2^2 Var(Y)$$

Likewise, $Var(X^*) = Var(b_1+b_2X) = b_2^2 Var(X)$

Therefore,

$$Corr(X^*,Y^*) = \frac{a_2b_2Cov(X,Y)}{\sqrt{a_2^2Var(Y)b_2^2Var(X)}} = \frac{a_2b_2Cov(X,Y)}{a_2b_2\sqrt{Var(Y)Var(X)}}$$

$$= Corr(X,Y)$$

Since $R^2 = r_{XY}^2$ and r_{XY} remains the same after a linear transformation, R^2 is unchanged.

(You could also use formulas for $Var(Y^*)$ and $Cov(X^*,Y^*)$)

(4)

29) $\hat{Y}_i = -0.4 + 1.0 X_i \Rightarrow \hat{\beta}_0 = -0.4$
$\quad\quad\quad\quad\quad (0.1) \quad\quad\quad\quad \hat{\beta}_1 = 1$

$$\hat{\beta}_1 = \frac{\sum x_t y_t}{\sum x_t^2} = 1 \Rightarrow \sum x_t y_t = \sum x_t^2$$

R^2 then becomes $\Rightarrow$ $$\frac{\left[\sum x_t y_t\right]^2}{\sum x_t^2 \sum y_t^2} = \frac{\left(\sum x_t^2\right)^{\cancel{2}}}{\cancel{\sum x_t^2} \sum y_t^2} = \frac{\sum x_t^2}{\sum y_t^2}$$

We also know that s.e. of $\hat{\beta}_1 = \sqrt{Var(\hat{\beta}_1)} = 0.1$ where $\hat{Var}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum x_t^2}$

Therefore, with $\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{N-k-1}$, we have:

$$\text{s.e. of } \hat{\beta}_1 = \sqrt{\frac{\sum \hat{u}_t^2}{\underset{52-1-1}{50}} \frac{1}{\sum x_t^2}} = 0.1$$

$$\Rightarrow \frac{\sum \hat{u}_t^2}{50 \sum x_t^2} = 0.01 \Rightarrow \sum x_t^2 = 2 \sum \hat{u}_t^2$$

Another representation of R^2 is:

$$R^2 = 1 - \frac{\sum \hat{u}_t^2}{\sum y_t^2} = \frac{\sum x_t^2}{\sum y_t^2}$$ (we found this above)

$$\Rightarrow \frac{\sum x_t^2 + \sum \hat{u}_t^2}{\sum y_t^2} = 1,$$ plugging in $\sum x_t^2 = 2 \sum \hat{u}_t^2$,

$$\frac{3 \sum \hat{u}_t^2}{\sum y_t^2} = 1 \Rightarrow \frac{\sum \hat{u}_t^2}{\sum y_t^2} = \frac{1}{3}$$

$$R^2 = 1 - \frac{\sum \hat{u}_t^2}{\sum y_t^2} = 1 - \frac{1}{3} = \frac{2}{3}$$