ECON 301 - REC #3

(PS #1)

Problem 1

a) $\hat{\beta}_1 = \dfrac{\sum x_t y_t}{\sum x_t^2} = \dfrac{-654}{60} = -10.9$

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 58 + 10.9 \times 5 = 112.5$

b) $R^2 = \hat{\beta}_1^2 \left(\dfrac{\sum x_t^2}{\sum y_t^2} \right) = (-10.9)^2 \dfrac{60}{7568} = 0.94$

c) $r_{XY} = \sqrt{R^2} \cong 0.97$

d) F test for whole significance $= \dfrac{R^2_u}{1-R^2_u} \cdot \dfrac{T-k-1}{p} \sim F^{\alpha}_{p, T-k-1}$

In our case $= \dfrac{0.94}{0.06} \times \dfrac{8}{2} \sim F^{0.05}_{2,8}$

$62 > 6.05 \rightarrow$ Reject H_0 ($H_0 = \beta_0 = \beta_1 = 0$)

The model is wholly significant.

e) First we need $se(\hat{\beta}_0)$

$R^2 = 1 - \dfrac{\sum \hat{u}_t^2}{\sum y_t^2} \rightarrow \sum \hat{u}_t^2 = (1-R^2) \sum y_t^2 = \underset{(1-0.94)}{0.06} \times 7568 \cong 454.08$

$\hat{\sigma}^2 = \dfrac{\sum \hat{u}_t^2}{T-k-1} = \dfrac{454.08}{8} = 56.76$

$\hat{Var}(\hat{\beta}_0) = \hat{\sigma}^2 \dfrac{\sum X_t^2}{T \sum x_t^2} = 56.76 \dfrac{310}{10 \times 60} = 29.326$

$se(\hat{\beta}_0) = \sqrt{Var(\hat{\beta}_0)} = \sqrt{29.326} = 5.41$

$\left.\begin{array}{l} H_0 = \beta_0 = 0 \\ H_A = \text{o.w.} \end{array}\right\}$ $t_c = \dfrac{\hat{\beta}_0 - 0}{se(\hat{\beta}_0)} = \dfrac{112.5}{5.41} = 20.79$

$t_{0.025, 8} = 2.306$

$t_c > t_{table} \Rightarrow$ Reject $H_0 \Rightarrow$ β_0 is significant, the model must have an intercept term

f) 99% CI for β_1:

$$Pr\{\hat{\beta}_1 - t_{0.005} \cdot se(\hat{\beta}_1) \leq \beta_1 \leq \hat{\beta}_1 + t_{0.005,8}\, se(\hat{\beta}_1)\} = 0.99$$

$$Var(\hat{\beta}_1) = \hat{\sigma}^2 \frac{1}{\sum x_t^2} = \frac{56.76}{60} = 0.946 \Rightarrow se(\hat{\beta}_1) = 0.9726$$

$$Pr\{-10.9 - \underbrace{3.355 \times 0.9726}_{3.26} \leq \beta_1 \leq -10.9 + \underbrace{3.355 \times 0.9726}_{3.26}\}$$

$$Pr\{-14.16 \leq \beta_1 \leq -7.64\} = 0.99 \Rightarrow 99\% \text{ CI for } \beta_1.$$

g) $\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t$, with $X_t = 15$:

$$\hat{Y}_t = 112.5 - 10.9 \times 15 = -51$$

h) CI for $E(Y)$: $\quad (E(Y) = \beta_0 + \beta_1 X_t,\ \hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t)$

$$Pr\{\hat{Y}_t - t_{0.025,8}\, \underset{\downarrow\ se(\hat{Y})}{\hat{\sigma}_{\hat{Y}}} \leq E(Y) \leq \hat{Y}_t + t_{0.025,8}\, \hat{\sigma}_{\hat{Y}}\} = 0.95$$

$\left(x_t = X_t - \bar{X} = 15 - 5\right)$

$$\text{se for predicted mean} = \hat{\sigma}_{\hat{Y}} = \sqrt{\hat{\sigma}^2\left[\underset{\downarrow\ \frac{1}{T}}{\cancel{\frac{1}{T}}} + \frac{x_t^2}{\sum x_t^2}\right]} = \sqrt{56.76\left[\frac{1}{10} + \frac{(15-5)^2}{60}\right]}$$

$$= 10.013$$

CI then becomes:

$$Pr\{-51 - \underset{\downarrow\ t_{0.025,8}}{2.306} \times 10.013 \leq E(Y) \leq -51 + 2.306 \times 10.013\}$$

$$\Rightarrow Pr\{-74.08 \leq E(Y) \leq -27.91\} = 0.95$$

CI for Y_t:

$$Pr\{\hat{Y}_t - t_{0.025,8}\,\hat{\sigma}_e \leq Y \leq \hat{Y}_t + t_{0.025,8}\,\hat{\sigma}_e\} = 0.95$$

se for predicted value = $\hat{\sigma}_e = \sqrt{\hat{\sigma}^2\left[1 + \frac{1}{T} + \frac{x_t^2}{\sum x_t^2}\right]}$

$$= \sqrt{56.76\left[1 + \frac{1}{10} + \frac{(15-5)^2}{60}\right]} = 12.53$$

$$Pr\{-51 - 2.306 \times 12.53 \leq Y \leq -51 + 2.306 \times 12.53\}$$

$$\Rightarrow Pr\{-79.89 \leq Y \leq -22.105\} = 0.95$$

i) $H_0 = \beta_0 + \beta_1 = 0$

$H_A = \beta_0 + \beta_1 \neq 0$

$$t_c = \frac{(\hat{\beta}_0 + \hat{\beta}_1) - 0}{\sqrt{Var(\hat{\beta}_0 + \hat{\beta}_1)}}$$

$$Var(\hat{\beta}_0 + \hat{\beta}_1) = Var(\hat{\beta}_0) + Var(\hat{\beta}_1) + 2\,Cov(\hat{\beta}_0, \hat{\beta}_1)$$

- $Var(\hat{\beta}_0) = 29.326$, $Var(\hat{\beta}_1) = 0.946$ (see parts (e) and (f))
- $Cov(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}\hat{\sigma}^2}{\sum x_t^2} = \frac{-5 \times 56.76}{60} = -4.73$

$Var(\hat{\beta}_0 + \hat{\beta}_1)$ then becomes $\Rightarrow 29.326 + 0.946 - 9.46 = 20.812$

$se(\hat{\beta}_0 + \hat{\beta}_1) = \sqrt{20.812} = 4.56$

$$t_c = \frac{\hat{\beta}_0 + \hat{\beta}_1}{se(\hat{\beta}_0 + \hat{\beta}_1)} = \frac{112.5 - 10.9}{4.56} = 22.28 = t_c > t_{table} = 2.306$$

Reject H_0 !

j) The model we have estimated is:

$$\hat{Y}_t = 112.5 - 10.9 X_t$$

When X_t is set to zero, the mean value of Y_t is 112,5. A one unit increase in X_t causes a 10,9 unit decline in Y_t. The model is wholly significant, as well as the intercept term. Finally, we can look at the significance of β_1:

For $H_0 = \beta_1 = 0$
$H_a = \beta_1 \neq 0$, $t_c = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)} = \frac{-10.9}{0.97} = -11.23$

0,97 ↓ found this in part (f)

$|t_c| > t_{table} = 2.306$

RH_0!

Therefore $\hat{\beta}_1$ is also individually significant.

k) We may keep the model as it is since it is wholly significant and all the coefficients are individually significant, along with a high R^2 value.

Problem 6

a) $C_t = b_0 + b_1 Y_t + u_t$

$$\hat{b}_1 = \frac{\sum c_t y_t}{\sum y_t^2} = \frac{\sum (C_t - \bar{C})(Y_t - \bar{Y})}{\sum (Y_t - \bar{Y})^2} = \frac{\sum C_t Y_t - \bar{Y}\sum C_t - \bar{C}\sum Y_t + N\bar{C}\bar{Y}}{\sum Y_t^2 + N\bar{Y}^2 - 2\bar{Y}\sum Y_t}$$

(with $T=10$, $\bar{Y} = \frac{\sum Y}{N} = 5$, $\bar{C} = \frac{\sum C}{N} = 4$)

$$\hat{b}_1 = \frac{236 - 5\times 40 - 4\times 50 + 10\times 5\times 4}{300 + 10\times 5^2 - 2\cdot 5\times 50} = \frac{36}{50} = 0.72$$

$$\hat{b}_0 = \bar{C} - \bar{Y}\hat{b}_1 = 4 - 5\times 0.72 = 0.4$$

$$r_{C,Y} = \frac{\sum c_t y_t}{\sqrt{\sum c_t^2 \sum y_t^2}} , \quad \sum c_t^2 = \sum (C_t - \bar{C})^2 = \sum C_t^2 + N\bar{C}^2 - 2\bar{C}\sum C_t$$

$$= 208.5 + 10\times 4^2 - 2\times 4\times 40 = 48.5$$

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$$r_{C,Y} = \frac{\sum c_t y_t}{\sqrt{\sum c_t^2 \sum y_t^2}} = \frac{36}{\sqrt{48.5 \times 50}} = \frac{36}{49.24} \cong 0.73$$

↳ we found $\sum y_t^2$ and $\sum c_t y_t$ when finding $\hat{b}_1$

b) $\hat{C} = \hat{b}_0 + \hat{b}_1 Y_t$

For the mean value of Y, we have:

[redacted] → $\hat{C} = 0.4 + 0.72 \times 5 = 4$

t value for df (degrees of freedom) $= 8$ $(T-k-1)$ and 5 % significance level is → 2,306

$$Pr\{\hat{C} - t_{0.025,8}\,\hat{\sigma}_e \leq C \leq \hat{C} + t_{0.025,8}\,\hat{\sigma}_e\} = 0.95$$

se for predicted value $= \hat{\sigma}_e = \sqrt{\hat{\sigma}^2\left[1 + \frac{1}{T} + \frac{y_t^2}{\sum y_t^2}\right]}$ $\quad y_t^2 = (5-5)^2 = 0$

$\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{T-k-1}$, $\sum \hat{u}_t^2 = (1-R^2)\sum c_t^2$ with $R^2 = r_{C,Y}^2 = 0.53$

$$\sum \hat{u}_t^2 = \underset{1-R^2}{0.47} \times 48.5 = 22.8 \Rightarrow \hat{\sigma}^2 = \frac{22.8}{8} = 2.85$$

Therefore, $\hat{\sigma}_e = \sqrt{2.85\left[1 + \frac{1}{10}\right]} \cong 1.77$

Plugging this value in CI, with $\hat{C} = 4$;

$$Pr\{4 - 2.306 \times 1.77 \leq C \leq 4 + 2.306 \times 1.77\}$$

$$= Pr\{-0.08 \leq C \leq 8.08\} = 0.95$$

c-d) MPC is given by $\hat{b}_1$ in $\hat{C} = \hat{b}_0 + \hat{b}_1 Y_t$, which is 0.72.

$H_0 = b_1 = 0.75$
$H_A = b_1 \neq 0.75$ $\Big\}$ $t_c = \frac{\hat{\beta}_1 - \beta_1}{se(\hat{\beta}_1)}$, $se(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{\sum y_t^2}} = \sqrt{\frac{2.85}{50}} = 0.23$

$$t_c = \frac{\overset{\to\, 0.72 - 0.75}{-0.03}}{0.23} = -0.13 \qquad |t_c| < t_{table} = 2.306 \to \text{DNR } H_0 \; !$$

e) For whole significance we can use F test:

$$\frac{R^2_u}{1-R^2_u} \cdot \frac{T-k-1}{p} \sim F^{\alpha}_{p, T-k-1}$$

$$\Rightarrow \frac{0.53}{0.47} \times \frac{8}{2} \sim F^{0.05}_{2,8} \Rightarrow$$

$4.51 > 4.45 \Rightarrow RH_0$ The model is wholly significant.

Problem 17:

a) For a_0:

$$Pr\{\hat{a}_0 - t_{0.005}\, se(\hat{a}_0) \leq a_0 \leq \hat{a}_0 + t_{0.005}\, se(\hat{a}_0)\} = 0.99$$

$\hat{a}_0 = 5.076$, $t_{0.005, 23} = 2.8$, $se(\hat{a}_0) = 2.006$

$$Pr\{5.076 - 2.8 \times 2.006 \leq a_0 \leq 5.076 + 2.8 \times 2.006\}$$

$$\Rightarrow Pr\{-0.54 \leq a_0 \leq 10.69\} = 0.99$$

For a_1:

$\hat{a}_1 = 0.345$, $se(\hat{a}_1) = 0.069$

$$Pr\{0.345 - 2.80 \times 0.069 \leq a_1 \leq 0.345 + 2.80 \times 0.069\}$$

$$\Rightarrow Pr\{0.15 \leq a_1 \leq 0.53\} = 0.99$$

b) t_t value at $\alpha = 0.05 = 2.06$ $(t_{0.025, 23})$

i) $H_0: a_0 = 0$
$H_A: a_0 \neq 0$ $\Rightarrow t_c = \dfrac{\hat{a}_0}{se(\hat{a}_0)} = \dfrac{5.076}{2.006} = 2.53 > 2.06 \rightarrow RH_0.$

$(|t_c| > t_{table})$

ii) $H_0: a_1 = 0$
$a_1 \neq 0$ $\Rightarrow t_c = \dfrac{\hat{a}_1}{se(\hat{a}_1)} = \dfrac{0.345}{0.069} = 5 > 2.06 \rightarrow RH_0.$

iii) $H_0: a_1 = a_0 = 0 \Rightarrow F\ test = \dfrac{R^2_u}{1-R^2_u} \cdot \dfrac{T-k-1}{p} = \dfrac{0.95}{0.05} \cdot \dfrac{23}{2} = 218.5$

$F_{calc} > F_{table}$ $(218.5 > F^{0.05}_{2,23} = 4.34) \Rightarrow RH_0.$

iv) $H_0: a_1 = 0.4$, $H_A: a_1 > 0.4$ } use $t_{0.05,23}$ this time (1.71) since we will use a one-tailed t test.

$$t_c = \frac{\hat{a}_1 - a_1}{se(\hat{a}_1)} = \frac{0.345 - 0.4}{0.069} = -0.79, \quad |-0.79| < 1.71$$

(t_c < t_{table})

DNR H_0.

! = There was a calculation error in this question when we solved it in the recitation hours.

v) $H_0 = a_1 = 4.5$
$H_A = a_1 < 4.5$

$$t_c = \frac{0.345 - 4.5}{0.069} \cong -64$$

$|-64| > 1.71$ - R H_0.

c) From (i) we know that a_0 is significant.
From (ii) " a_1 is significant.
From (iii) " the model is wholly significant.
Therefore, we do not need a reformulation of the model.

d) See problem 1-h.